ALGEBRAIC NUMBER THEORY FINAL EXAM

Attempt all questions. Total : 100 marks

(1) Let $K = \mathbb{Q}[\sqrt{2}, \sqrt{3}]$. Note that K is a normal extension of $\mathbb{Q}$ of degree 4, with Galois group $\mathrm{Gal}(K/\mathbb{Q})$ equal to the Klein 4 group. Also note that K contains 3 quadratic subfields $\mathbb{Q}[\sqrt{2}]$, $\mathbb{Q}[\sqrt{3}]$ and $\mathbb{Q}[\sqrt{6}]$.
   - (a) Let p be a prime in $\mathbb{Z}$ which is ramified in each of the quadratic subfields. What happens to p in K? Give an example of such a prime p.(20 marks)
   - (b) Let p be a prime in $\mathbb{Z}$ which splits completely in each of the quadratic subfields. What happens to p in K? Give an example of such a prime p.(20 marks)

(2) Let $K \subset L \subset M$ be a tower of number fields, such that M and L are both normal extensions of K. Let P be a prime of K, let Q be a prime of L lying above P, let U be a prime of M lying above Q. Assume, P is unramified in M (and, hence in L). We thus have the Frobenius automorphisms $\phi(U|P)$, $\phi(U|Q)$ and $\phi(Q|P)$. Prove that $\phi(U|Q) = \phi(U|P)^{f(Q|P)}$, and $\phi(Q|P)$ is the restriction of $\phi(U|P)$ to L (here, $f(Q|P)$ is the inertia degree of Q over P). (10+10 = 20 marks)

(3) Compute the ideal class group of the number ring of $\mathbb{Q}[\sqrt{-14}]$. You may wish to use the result that, given any element of the class group of a number ring R, there exists an ideal J representing this element, such that $||J|| \leq \frac{n!}{n^n}(\frac{4}{\pi})^s\sqrt{|disc(R)|}$. (20 marks)

(4) State the Dirichlet Unit Theorem. Compute the fundamental unit and the unit group of the number ring $\mathbb{Z}[\sqrt{11}]$. (10+10 = 20 marks)